

# Proof of a conjecture on the zero forcing number of a graph \*

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**Abstract** Amos et al. (Discrete Appl. Math. 181 (2015) 1-10) introduced the notion of the  $k$ -forcing number of graph for a positive integer  $k$  as the generalization of the zero forcing number of a graph. The  $k$ -forcing number of a simple graph  $G$ , denoted by  $F_k(G)$ , is the minimum number of vertices that need to be initially colored so that all vertices eventually become colored during the discrete dynamical process by the following rule. Starting from an initial set of colored vertices and stopping when all vertices are colored: if a colored vertex has at most  $k$  non-colored neighbors, then each of its non-colored neighbors become colored. Particular,  $F_1(G)$  is a widely studied invariant with close connection to the maximum nullity of a graph, under the name of the zero forcing number, denoted by  $Z(G)$ . Among other things, the authors proved that for a connected graph  $G$  of order  $n$  with  $\Delta = \Delta(G) \geq 2$ ,  $Z(G) \leq \frac{(\Delta-2)n+2}{\Delta-1}$ , and this inequality is sharp. Moreover, they conjectured that  $Z(G) = \frac{(\Delta-2)n+2}{\Delta-1}$  if and only if  $G = C_n$ ,  $G = K_{\Delta+1}$  or  $G = K_{\Delta,\Delta}$ . In this note, we show the above conjecture is true.

**Keywords:** Zero forcing set; Zero forcing number; Rank; Nullity

## 1 Introduction

We consider undirected finite simple connected graphs only. For notation and terminology not defined here, we refer to [6]. For a graph  $G = (V(G), E(G))$ ,  $|V(G)|$  and  $|E(G)|$  are its order and size, respectively. For a vertex  $v \in V(G)$ , the neighborhood  $N(v)$  of  $v$  is defined as the set of vertices adjacent to  $v$ . The degree  $d_G(v)$  of  $v$  is the number of edges incident with  $v$  in  $G$ . The minimum and maximum degrees of a vertex in a graph  $G$  are denoted  $\delta(G)$  and  $\Delta(G)$ , respectively. Let  $S \subseteq V(G)$ . Denote the set of the edges between  $S$  and  $\bar{S}$  by  $E(S, \bar{S})$ , and let  $e(S, \bar{S}) = |E(S, \bar{S})|$ . The subgraph induced by  $S$ , denoted by

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\*Research supported by NSFC (No. 11161046) and by Xingjiang Talent Youth Project (No. 2013721012).

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$G[S]$ , is the graph with vertex set  $S$ , in which two vertices  $x$  and  $y$  are adjacent if and only if they are adjacent in  $G$ . As usual, for a positive integer  $n \geq 1$ ,  $K_n$  and  $K_{n,n}$  denote respectively the complete graph of order  $n$  and the complete bipartite graph with  $n$  vertices in its each part;  $C_m$  denote the cycle of order  $m$  for an integer  $m \geq 3$ .

Next, we follow the definition by Amos et al. [2]. Let  $k$  be a positive integer and  $G$  a graph. A set  $S \subseteq V(G)$  is a  $k$ -forcing set if, when its vertices are initially colored - while the remaining vertices are initially non-colored - and the graph is subjected to the following color change rule, all of vertices in  $G$  will eventually become colored. A colored vertex with at most  $k$  non-colored neighbors will cause each the non-colored neighbor to become colored. The  $k$ -forcing number of  $G$ , denoted by  $F_k(G)$ , is the cardinality of the smallest  $k$ -forcing set. If a vertex  $u$  cause a vertex  $v$  change colors during the  $k$ -forcing process, we say that  $u$   $k$ -forces  $v$  (in particular,  $u$  forces  $v$  when  $k = 1$ ).

This concept generalizes a widely studied notion of the zero forcing number  $Z(G)$  of a graph  $G$ . Indeed,  $F_1(G) = Z(G)$ . Barioli et al. [3] and Burgarth et al. [7] introduced independently the concepts of zero forcing set and zero forcing number of a graph. In [3], it is introduced to bound the maximum nullity  $M(G)$  of a graph. Namely, for a graph  $G$  whose vertices are labeled from 1 to  $n$ ,  $M(G)$  denote the maximum nullity over all symmetric real valued matrices where, for  $i \neq j$ , the  $ij$ th entry in nonzero if and only if  $ij$  is an edge in  $G$ . Then,  $M(G) \leq Z(G)$  for any graph  $G$ . For the more results on the relation between the relation of the maximum nullity and the zero forcing number of a graph, we refer to [4, 5, 9, 10, 11, 12, 13, 14, 15]. In [7], the zero forcing set of a graph has been used in order to study the controllability of quantum systems. Aazami [1] proved the NP-hardness of computing the zero forcing number of a graph, using a reduction from the Directed Hamiltonian Cycle problem.

Amos et al. [2] generalized the concept of zero forcing number of a graph to the  $k$ -forcing number of a graph for an integer  $k \geq 1$  and proved that for a connected graph  $G$  of order  $n$  with  $\Delta = \Delta(G) \geq 2$ ,  $Z(G) \leq \frac{(\Delta-2)n+2}{\Delta-1}$ , and this inequality is sharp. Moreover, they posed the following conjecture.

**Conjecture (Amos et al. [2]).** Let  $G$  be a connected graph with  $\Delta \geq 2$ . Then

$$Z(G) = \frac{(\Delta-2)n+2}{\Delta-1},$$

if and only if  $G = C_n$ ,  $G = K_{\Delta+1}$  or  $G = K_{\Delta,\Delta}$ .

In this note, we confirm the validity of the above conjecture.

## 2 Some results on $Z(G)$

A  $k$ -dominating set of a graph  $G$  is a set  $D$  of vertices such that every vertex not in  $D$  is adjacent to at least  $k$  vertices in  $D$ .

**Lemma 2.1.** (*Lemma 4.1 in [2]*) Let  $k$  be a positive integer and  $G = (V, E)$  be a  $k$ -connected graph with  $n > k$ . If  $S$  is a smallest  $k$ -forcing set such that the subgraph induced by  $V \setminus S$  is connected, then  $V \setminus S$  is a connected  $k$ -dominating set of  $G$ .

**Theorem 2.2.** ([2]) Let  $k$  be positive integer and let  $G = (V, E)$  be a  $k$ -connected graph with  $n > k$  vertices and  $\Delta \geq 2$ . Then

$$F_k(G) \leq \frac{(\Delta - 2)n + 2}{\Delta + k - 2},$$

and this inequality is sharp.

**Theorem 2.3.** (Corollary 3.1 in [8]) Let  $G$  be a connected graph of order  $n$  with maximum degree  $\Delta$  and minimum degree  $\delta$ . Then

$$Z(G) \leq \frac{(\Delta - 2)n - (\Delta - \delta) + 2}{\Delta - 1}.$$

**Lemma 2.4.** Let  $T$  be a tree with exactly  $k$  leaves. If  $S$  is a set of  $k - 1$  leaves of  $T$ , then  $S$  is a zero forcing set of  $T$ .

*Proof.* The proof is by induction on  $k$ . If  $k = 2$ ,  $T$  is path, and the result clearly holds. Now assume that  $k \geq 3$ . Take a vertex  $u \in S$ . Let  $P$  be a maximal path of  $T$  containing  $u$  such that every vertex  $v$  on  $P$  has degree at most two in  $T$ . Let  $T' = T - V(P)$ . Note that  $T'$  has exactly  $k - 1$  leaves. By the induction hypothesis,  $S' = S \setminus \{u\}$  is a zero forcing set of  $T'$ . So,  $S$  is a zero forcing set of  $T$ .  $\square$

### 3 Main result

**Theorem 3.1.** Let  $G$  be a connected graph with  $\Delta \geq 2$ . Then

$$Z(G) = \frac{(\Delta - 2)n + 2}{\Delta - 1},$$

if and only if  $G = C_n$ ,  $G = K_{\Delta+1}$  or  $G = K_{\Delta, \Delta}$ .

*Proof.* It is clear that  $Z(C_n) = 2$  for any  $n \geq 3$ ,  $Z(K_{\Delta+1}) = \Delta$ ,  $Z(K_{\Delta, \Delta}) = 2\Delta - 2$ . Hence, the sufficiency of theorem holds trivially.

To show the necessity, we assume that  $G$  is a connected graph of order  $n$  with  $\Delta \geq 2$  and  $Z(G) = \frac{(\Delta - 2)n + 2}{\Delta - 1}$ . By Theorem 2.3,  $G$  is a  $\Delta$ -regular graph. If  $\Delta = 2$ , then  $G = C_n$ . In what follows, we assume that  $\Delta \geq 3$ .

Let  $S$  be a smallest zero forcing set of  $G$  such that  $G[\overline{S}]$  is connected, where  $\overline{S} = V \setminus S$ . Thus,

$$|S| \geq Z(G) = \frac{(\Delta - 2)n + 2}{\Delta - 1}. \quad (1)$$

**Claim 1.** Each vertex of  $S$  has exactly one neighbor in  $\overline{S}$  and  $G[\overline{S}]$  is a tree.

*Proof.* By Lemma 2.1,

$$e(S, \overline{S}) \geq |S|. \quad (2)$$

On the other hand,

$$\begin{aligned} e(S, \overline{S}) &= \sum_{v \in \overline{S}} (d(v) - d_{\overline{S}}(v)) \\ &= \sum_{v \in \overline{S}} d(v) - \sum_{v \in \overline{S}} d_{\overline{S}}(v) \\ &\leq \Delta |\overline{S}| - 2(|\overline{S}| - 1) \\ &= (\Delta - 2)|\overline{S}| + 2 \\ &= (\Delta - 2)(n - |S|) + 2. \end{aligned} \quad (3)$$

Combining (2) and (3), we have

$$|S| \leq \frac{(\Delta - 2)n + 2}{\Delta - 1}. \quad (4)$$

Combining (1) and (4), we have

$$|S| = \frac{(\Delta - 2)n + 2}{\Delta - 1}. \quad (5)$$

From (5), we can conclude that  $S$  is a smallest forcing set of  $G$  and that each vertex of  $S$  has exactly one neighbor in  $\overline{S}$  and  $G[\overline{S}]$  is a tree.  $\square$

Note that

$$|\overline{S}| = n - |S| = \frac{n - 2}{\Delta - 1}. \quad (6)$$

If  $|\overline{S}| = 1$ , by (6),  $\Delta = n - 1$ . Since  $G$  is  $(n - 1)$ -regular,  $G \cong K_n = K_{\Delta+1}$ . Next we assume that  $|\overline{S}| \geq 2$  and let  $x$  be a leaf of  $G[\overline{S}]$  and  $X = N(x) \cap S = \{x_1, \dots, x_{\Delta-1}\}$ .

**Claim 2.**  $X$  is either an independent set or a clique.

*Proof.* We assume that  $X$  is not an independent set, and show that  $X$  is a clique. Let  $x_1, x_2 \in X$  with  $x_1 x_2 \in E(G)$ . Since  $\Delta \geq 3$ , there exists a neighbor  $y_1$  of  $x_1$  in  $S \setminus X$ .

First we show that  $y_1$  is adjacent to all vertices of  $X$  in  $G$ . To see this, suppose that there exists a vertex  $x_j \in X$ , where  $2 \leq j \leq \Delta - 1$ , which is not adjacent to  $y_1$ . Since  $\Delta \geq 3$ , by Claim 1, there exists a neighbor  $y_j \in S \setminus X$  of  $x_j$  in  $G$ . Set  $S' = S \cup \{x\} \setminus \{y_1, y_j\}$ . We can show that  $S'$  is a zero forcing set of  $G$ . Observe that all neighbors of  $x_j$  but  $y_j$  are initially colored. So, by the color exchange rule,  $y_j$  should be colored. Now, all neighbors of  $x_1$  but  $y_1$  are colored. By the color

exchange rule,  $y_1$  is forced to be colored. All vertices of  $S$  are colored, and thus  $S'$  is a zero forcing set of  $G$ . Since  $|S'| < |S|$ , which contradicts the fact that  $S$  is a minimum zero forcing set of  $G$ .

Next we show that  $x_1$  is adjacent to all vertices of  $X$  in  $G$ . Suppose that this is not, and that  $x_1x_j \notin E(G)$  for some vertex  $x_j \in X$ . Set  $S' = S \cup \{x\} \setminus \{x_1, y_1\}$ . We consider  $x_j$ . Note that all neighbors of  $x_j$  but  $y_1$  are initially colored. By the color exchange rule,  $y_1$  is colored. Now, all neighbors of  $x_2$  but  $x_1$  are colored. By the color exchange rule,  $x_1$  is colored. Since  $|S'| < |S|$ , which contradicts the fact that  $S$  is a minimum zero forcing set of  $G$ .

Finally, by an argument similar to the above, one can prove that  $x_i$  is adjacent to every other vertex in  $X$  for each  $i \geq 2$ . Thus,  $X$  is a clique of  $G$ .

This proves the claim.  $\square$

**Claim 3.** If  $X$  is a clique of  $G$ , then there exists a unique vertex  $y$  in  $S$  such that  $N(y) = X \cup \{z\}$ , and  $d_{G[\overline{S}]}(z) = \Delta - 1$ , where  $z$  is the unique neighbor of  $y$  in  $\overline{S}$ .

*Proof.* The first half of the assertion can be deduced from the proof of Claim 2 (see the paragraph starting with “First we show”). We show  $d_{G[\overline{S}]}(z) = \Delta - 1$  by contradiction. Suppose that  $d_{G[\overline{S}]}(z) \neq \Delta - 1$ , and let  $z' \in N(z) \cap S$  and  $z'' \in N(z') \cap S$ . Note that  $y \neq z'$  and  $y \neq z''$ . Set  $S' = S \cup \{z\} \setminus \{z'', x_1\}$ . By the color exchange rule,  $z'$  forces  $z''$ , and  $y$  forces  $x_1$ . Now all vertices of  $S$  are already colored. But,  $|S'| < |S|$ , a contradiction.  $\square$

**Claim 4.** If  $X$  is an independent set of  $G$ , then  $N(x_i) \cap S = N(x_j) \cap S$  for any two vertices  $x_i, x_j \in X$ , and  $N(x_i) \cap S$  is an independent set of  $G$  with cardinality  $\Delta - 1$ . Moreover, if  $z_i$  is a leaf of  $G[\overline{S}]$ , where  $z_i$  is the unique neighbor of  $y_i \in N(x_i) \cap S$ , then  $N(N(x_i) \cap S) \cap \overline{S} = \{z_i\}$ .

*Proof.* By an argument similar to the proof of Claim 2 (see the paragraph starting with “First we show”), one can show that  $N(x_i) \cap S = N(x_j) \cap S$  for any two vertices  $x_i, x_j \in N(x) \cap S$ . By contradiction, suppose that  $y_j \in N(x_i) \cap S$  is not adjacent to  $z_i$  in  $G$ . Since  $z_i$  is a leaf of  $G[\overline{S}]$  and  $G$  is  $\Delta$ -regular,  $z_i$  has a neighbor  $z' \in S \setminus (X \cup N(X))$  and  $z'' \in N(z') \cap S$ . Note that  $z'' \in S \setminus (X \cup N(X))$ . Set  $S' = S \cup \{z_i\} \setminus \{z'', x_1\}$ . By the color exchange rule,  $z'$  forces  $z''$ , and then  $y_j$  forces  $x_1$ . Now all vertices of  $S$  are already colored. But,  $|S'| < |S|$ , a contradiction.  $\square$

Before proceeding, we recall the definition of bridge, which can be found on the page 263 in [6]. Let  $H$  be a proper subgraph of a connected graph  $G$ . The set  $E(G) \setminus E(H)$  may be partitioned into classes as follows.

(i). For each component  $F$  of  $G - V(H)$ , there is a class consisting of the edges of  $F$  together with the edges linking  $F$  to  $H$ .

(ii). Each remaining edge  $e$  (that is, one which has both ends in  $V(H)$ ) defines a singleton class  $\{e\}$ .

The subgraphs of  $G$  induced by these classes are the *bridges* of  $H$  in  $G$ . For a bridge  $B$  of  $H$ , the elements of  $V(B) \cap V(H)$  are called its vertices of attachment to  $H$ ; the remaining vertices of  $B$  are its internal vertices. A bridge is trivial if it has not internal vertices. A bridge with  $k$  vertices of attachment is called a  $k$ -bridge. Observe that bridges of  $H$  can intersect only in vertices of  $H$ .

**Claim 5.** Let  $B_i$  be a bridge of  $G[\overline{S}]$  containing a leaf  $z_i$  of  $G[\overline{S}]$  for  $1 \leq i \leq 2$ . Then  $B_1 = B_2$  or  $V(B_1) \cap V(B_2) = \emptyset$ .

*Proof.* By contradiction, suppose that  $B_1 \neq B_2$  and  $V(B_1) \cap V(B_2) \neq \emptyset$ , and let  $w \in V(B_1) \cap V(B_2)$ . Let  $w_1 \in V(B_1) \cap S$  and  $w_2 \in V(B_2) \cap S$ . Take a vertex  $w'_1 \in N(w_1) \cap S$  and a vertex  $w'_2 \in N(w_2) \cap S$ . Set  $S' = S \cup \{w\} \setminus \{w'_1, w'_2\}$ . In this case,  $w_1$  forces  $w'_1$  and  $w_2$  forces  $w'_2$ . Thus, all vertices of  $S$  are colored. This shows that  $S'$  is zero forcing set of  $G$ , a contradiction.  $\square$

We consider the case when  $|\overline{S}| = 2$ . Let  $\overline{S} = \{z_1, z_2\}$ . Let  $B_i$  be the bridge of  $G[\overline{S}]$  containing  $z_i$  for  $1 \leq i \leq 2$ . Since  $|V(B_i) \cap \overline{S}| \geq 2$  and  $|\overline{S}| = 2$ , by Claim 5,  $B_1 = B_2$ , which implies that  $G \cong K_{\Delta, \Delta}$ .

Next, we complete the proof by showing that  $|\overline{S}| \geq 3$  is not possible. We consider the following cases.

**Case 1.**  $X$  is a clique of  $G$ .

Let  $S' = S \setminus \{x_1\}$ . We will show that  $S'$  is a zero forcing set of  $G$ . By Claim 5, each leaf  $z$  of  $G[\overline{S}]$  distinct from  $x$  is forced to be colored in 1 by some vertex in  $S'$ . Note that  $T = G[\overline{S} \cup \{x_1\}]$  is a tree with exactly  $k$  leaves. By Lemma 2.4,  $L \setminus \{x_1\}$  is a zero forcing set of  $T$ , where  $L$  is the set of leaves  $G[\overline{S}]$ . This shows that  $S'$  is a zero forcing set of  $G$ , contradicting the choice of  $S$ .

**Case 2.**  $N(x) \cap S$  is an independent set of  $G$  for each leaf  $x$  of  $G[\overline{S}]$ .

Take a leaf  $x$  of  $G[\overline{S}]$ , and let  $N(x) \cap S = \{x_1, \dots, x_{\Delta-1}\}$ . By Claim 3, we know that  $N(x_i) \cap S = N(x_j) \cap S$  for any two neighbors  $x_i, x_j \in S$  in  $G$ , and  $N(x_i) \cap S$  is an independent set of  $G$  with cardinality  $\Delta - 1$ . Let  $N(x_i) \cap S = \{y_1, \dots, y_{\Delta-1}\}$ . Let  $z_i \in \overline{S}$  be the unique neighbor of  $y_i$  in  $G$ . By Claim 4, we consider two subcases.

**Case 2.1.**  $z_i$  is not a leaf of  $G[\overline{S}]$  for each  $i \in \{1, \dots, \Delta - 1\}$ .

Let  $S' = S \setminus \{x_1\}$ . By an argument same as the proof of tackling Case 1, one may obtain a contradiction by showing that  $S'$  is a zero forcing set of  $G$ .

**Case 2.2.**  $z_i$  is a leaf of  $G[\overline{S}]$  and  $z_j = z_i$  for each  $j$  other than  $i$ .

For the simplicity, let  $z = z_i$ . Let  $S' = S \setminus \{x_1\}$ . Since  $N(x) \cap S$  is an independent set of  $G$ ,  $x$  is forced to colored in 1 by  $x_2$ . Note that  $S'$  forces to color all leaves of  $G[\overline{S}]$  but  $z$ . Let  $u$  be the neighbor of  $x$  in  $G[\overline{S}]$ . If  $u$  has a

neighbor  $u'$  in  $S$ , by Claim 5, then  $u'$  is neither a  $x_i$  nor a  $y_j$ . So,  $u$  is forced to be colored in 1 by  $u'$ . Then,  $x$  forces  $x_1$ . Now, all vertices in  $S$  are colored in 1. So,  $S'$  is a zero forcing set of  $G$  and  $|S'| < |S|$ , which contradicts the fact that  $S$  is a minimum zero forcing set of  $G$ . Now we assume that  $u$  has no neighbor in  $S$ . Hence,  $d_{G[\overline{S}]}(u) = \Delta \geq 3$  and the number of leaves of  $G[\overline{S}] - x$  is  $k - 1$ . By the color exchange rule, all leaves of  $G[\overline{S}] - x$  but  $z$  are forced to be colored in 1 by  $S'$ . By Lemma 2.4, all vertices in  $\overline{S} \setminus \{x\}$  will be forced to be colored in 1, and then  $x$  forces  $x_1$ . This shows that  $S'$  is a zero forcing set of  $G$ , contradicting the choice of  $S$ .

This completes the proof. □

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